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# A Sixth-Order Family of Methods for Nonlinear Equations

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A one-parameter family of sixth-order methods for finding simple zeros of nonlinear functions is developed. Each member of the family requires three evaluations of the given function and only one evaluation of the derivative per step.

## 1. INTRODUCTION

Newton's method for computing a simple zero $\xi$ of a nonlinear equation $f(x)=0$ has been modified in a number of ways. For example, Ostrowski [1] discusses a third-order method that evaluates the function $f$ at every substep but only requires the derivative $f'$ at every other substep. He also introduced a fourth-order scheme that uses the same information. King [2] has shown that there is a family of such methods. Traub [3] introduced a third-order method which requires one function and two derivative evaluation per step. Jarratt [4] developed fourth-order method which uses the same information. King [5] developed a fifth-order scheme that requires two evaluations of $f$ and $f'$.

Here we develop methods of order six. An iteration consists of one Newton substep followed by two substeps of "modified" Newton (i.e., using the derivative of $f$ at the first substep instead of the current one).

Let us recall the definition of order (see e.g. [3]).

DEFINITION Let $x_1, x_2, \ldots, x_i$, be a sequence converging to $\xi$. Let

$$\varepsilon_i = x_i - \xi.$$

If there exists a real number $p$ and a nonzero constant $C$ such that

$$\frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} \to C \tag{2}$$

then $p$ is called the order of the sequence.

## 2. DEVELOPMENT OF THE SIXTH-ORDER FAMILY

Development of the sixth-order family.

Let

$$
\left.
\begin{aligned}
\omega_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
z_n &= \omega_n - \frac{f(\omega_n)}{f'(x_n)} \cdot \frac{f(x_n) + A f(\omega_n)}{f(x_n) + B f(\omega_n)} \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \cdot \frac{f(x_n) + C f(\omega_n) + D f(z_n)}{f(x_n) + G f(\omega_n) + H f(z_n)}
\end{aligned}
\right\} \tag{3}
$$

where $A$, $B$, $C$, $D$, $G$, $H$ are arbitrary constants. This is a family of methods which uses Newton's method in the first substep and two Newton-like in the other two substeps. In each step we have to evaluate the function $f(x)$ at the three points $x_n$, $\omega_n$, $z_n$ and to evaluate the derivative at one point $x_n$.

In order to find the order of the method we used MACSYMA (Project MAC's SYmbolic MAnipulation system, which is a large computer programming system written in LISP and used for performing symbolic as well as numerical mathematical manipulations [6]).

The error expression at $\omega_n$ is given by

$$
\begin{aligned}
\varepsilon(\omega_n) = {} & \tfrac{1}{2}F_2\varepsilon_n^2 + \tfrac{1}{6}(2F_3 - 3F_2^2)\varepsilon_n^3 + \tfrac{1}{24}(-14F_2F_3 + 12F_2^3 + 3F_4)\varepsilon_n^4 \\
& + \tfrac{1}{120}(-20F_2^2 + 100F_2^2F_3 - 25F_2F_4 - 60F_2^4 + 4F_5)\varepsilon_n^5 \\
& + \tfrac{1}{720}(5F_6 - 39F_5F_2 - 85F_4F_3 + 210F_4F_2^2 + 330F_2F_3^2 - 780F_3F_2^3 \\
& + 360F_2^5)\varepsilon_n^6 + \dots
\end{aligned} \tag{4}
$$

where

$$
F_i = \frac{f^{(i)}(\xi)}{f'(\xi)} \tag{5}
$$

and

$$
\varepsilon_n = \varepsilon(x_n) = x_n - \xi. \tag{6}
$$

For later use we also give the expression for

$$
\begin{aligned}
f(\omega_n) = {} & (\tfrac{1}{2}F_2\varepsilon_n^2 + \tfrac{1}{6}(F_3F_1 - 3F_2^3)\varepsilon_n^3 + \tfrac{1}{24}(3F_4 - 14F_2F_3 + 15F_2^3)\varepsilon_n^4 \\
& + \tfrac{1}{120}(4F_5 - 25F_2F_4 - 20F_3^2 + 120F_2^2F_3 - 90F_2^4)\varepsilon_n^5
\end{aligned}
$$

$$+\tfrac{1}{720}(5F_6-39F_5F_2-85F_4F_3+255F_4F_2^2+370F_3^2F_2$$
$$-1095F_3F_2^3+630F_2^5)\varepsilon_4^6+\dots\}f'(\xi) \tag{7}$$

and

$$\begin{aligned}f'(x_n)[f(x_n)+Bf(\omega_n)] &= f'^2(\xi)\{\varepsilon_n+\tfrac{1}{2}(B+3)F_2\varepsilon_n^2\\ &+\tfrac{1}{6}[(2B+4)F_3+3F_2^2]\varepsilon_n^3\\ &+\tfrac{1}{24}[(3B+5)F_4+10F_2F_3+3BF_2^3]\varepsilon_n^4\\ &+\tfrac{1}{120}[(4B+6)F_5+15F_2F_4+10F_3^2+20BF_2^2F_3\\ &-15BF_2^4]\varepsilon_4^5\\ &+\tfrac{1}{720}[(5B+7)F_6+21F_5F_2+35F_4F_3+45BF_4F_2^2\\ &+40BF_2F_3^2-150BF_3F_2^3+90BF_2^5]\varepsilon_n^6+\dots\end{aligned} \tag{8}$$

The error expression at the point $z_n$ is given by

$$\begin{aligned}\varepsilon(z_n) &= \tfrac{1}{4}(B-A+2)F_2^2\varepsilon_n^3+\tfrac{1}{24}[(8B-8A+14)F_2F_3+(-3B^2\\ &+(3A-21)B+21A-27)F_2^3]\varepsilon_n^4+\tfrac{1}{144}[(18B-18A+30)F_2F_4\\ &+(16B-16A+24)F_3^2+(-36B^2+(36A-228)B+228A\\ &-264)F_2^2F_3+(9B^3+90B^2-9AB^2-90AB+297B-297A\\ &+270)F_2^4]\varepsilon_n^5+\tfrac{1}{1440}[(48B-48A+78)F_2F_5+(120B-12A\\ &+170)F_3F_4+(-135B+135AB-825B+825A-930)F_2^2F_4\\ &+(-240B^2+240AB-1360B+1360A-1400)F_2F_3^2+(240B^3\\ &-240AB^2+2220B^2-2220AB+6750B-6750A+5640)F_2^3F_3\\ &+(-45B^4+45AB^3-585B^3+585AB^2-2790B^2+2790AB\\ &-5805B+5805A-3960)F_2^5]\varepsilon_n^6+\dots\}\end{aligned} \tag{9}$$

In order to annihilate the coefficient of $\varepsilon_n^3$ in the expression for $\varepsilon(z_n)$ we have chosen $B=A-2$. We have also chosen $c=-1$, $G=-3$ and $H=D$ so that the low-order terms in the expression for $\varepsilon_{n+1}$ will vanish. This choice leads to the family

$$\left.\begin{aligned} \omega_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= \omega_n - \frac{f(\omega_n)}{f'(x_n)} \frac{f(x_n) + Af(\omega_n)}{f(x_n) + (A-2)f(\omega_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{f(x_n) - f(\omega_n) + Df(z_n)}{f(x_n) - 3f(\omega_n) + Df(z_n)} \end{aligned}\right\} \tag{10}$$

with error term

$$\varepsilon_{n+1} = \tfrac{1}{144}[2F_3^2F_2 - 3(2A+1)F_2^3F_3]\varepsilon_n^6 + \ldots \tag{11}$$

Note that the error term does not depend on $D$ so we can let $D=0$.

If the parameter $A$ is chosen so that $A = -\frac{1}{2}$ then the $F_2^3F_3$ term in error is eliminated so that

$$\left.\begin{aligned} \omega_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= \omega_n - \frac{f(\omega_n)}{f'(x_n)} \frac{f(x_n) - \frac{1}{2}f(\omega_n)}{f(x_n) - \frac{5}{2}f(\omega_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{f(x_n) - f(\omega_n)}{f(x_n) - 3f(\omega_n)} \end{aligned}\right\} \tag{12}$$

and

$$\varepsilon_{n+1} = \tfrac{1}{72}F_3^2F_2\varepsilon_n^6 + \ldots \tag{13}$$

## 3. NUMERICAL EXAMPLES

Computer tests for the functions $f = x^n - 1$ with the root $\xi = 1$ for various values of the integer $n$ show that the method is indeed of order 6.

*Remark* Our method needs the same number of function evaluations as a method constructed from one substep of Newton's method followed by two substeps of the secant method. The order of the secant method is approximately 1.62 (see e.g. [7]) thus such a scheme will approximately be of order 5.2 (see e.g. [4]).

## References

[1] A. M. Ostrowski, *Solution of Equations and Systems of Equations*, Academic Press, New York, 1960.
[2] Richard F. King, A family of fourth-order methods for nonlinear equations, submitted to *Siam J. Numer. Anal.*

[3] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.
[4] P. Jarratt, Some efficient fourth-order multipoint methods for solving equations, *BIT* **9** (1969), 119–124.
[5] Richard F. King, A fifth-order family of modified Newton methods, *BIT* **11** (1971), 409–412.
[6] R. Bogen *et al.*, *MACSYMA Reference Manual*, MIT, Cambridge, Mass., 1975.
[7] D. M. Young and R. T. Gregory, *A Survey of Numerical Mathematics*, Addison-Wesley, Reading, Mass., 1973.